GOSTS Descriptive Set Theory
Topological Preliminaries and the Boldface Hierarchies

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There are many ways to characterize \( \mathbb{R} \) and many more constructions to arrive at them. For example, we can take \( \mathbb{R} \) as

- the unique Archimedean, complete ordered field;
- the unique separable, complete, dense linear order without endpoints;
- Dedekind cuts of \( \mathbb{Q} \);
- the completion of \( \mathbb{Q} \) via equivalence classes of Cauchy sequences;
- and many more.

We will care primarily about \( \mathbb{R} \)'s topology as a metric space, but also similar topologies that are slightly nicer but still translate easily to the topology on \( \mathbb{R} \).
What is a Polish space?

This primarily means

- The Baire Space - $\mathcal{N} = \omega \omega$; and
- The Cantor Space - $\mathcal{C} = \omega^2$.

These are nice in that they can be examined with trees: $\mathcal{N}$ consists of the infinite branches of $<^\omega \omega$, and $\mathcal{C}$ of $<^\omega 2$. Beyond these we also care about

- The natural numbers $\omega$;
- (Countable) products of the above.

How do we encapsulate all of these spaces? We have the concept of a Polish space.
What is a Polish space?

**Definition**

- A **polish metric space** is a complete metric space $\langle X, d \rangle$ that is separable (i.e. has a countable dense subset).

We have lots and lots of examples of complete metric spaces just because every metric space has a completion.

- $\mathbb{R}$ with the usual metric $d(x, y) = |x - y|$ is complete.
- Every set $X$ with the discrete metric is complete ($d(x, y) = 1$ if $x \neq y$ otherwise $0$).
- Every metric space $\langle X, d \rangle$ has a completion, i.e. a $\langle Y, d^+ \rangle$ where $X \subseteq Y$ and $d^+ |_{X \times X} = d$. 
Definition

- A *polish metric space* is a complete metric space \( \langle X, d \rangle \) that is separable (i.e. has a countable dense subset).
- A *polish topological space* (or just a *polish space*) is a topology derived from a polish metric space.

The topology derived from a metric space is just the topology where we set open balls to be open. Note that there can be multiple different metrics giving the same topology: \( d_1(x, y) = 2|x - y| \) and \( d_2(x, y) = |x - y| \) give the same topology on \( \mathbb{R} \), for example. So it’s more important to really look at the topology rather than the metric itself.
What is a Polish space?

**Definition**

A *polish space* is the topology on a separable, complete metric space.

Some examples of Polish spaces are the following

- $\mathbb{R}$ with the usual topology is polish.
- The discrete topology on any countable set is a polish space.
- If $X$ and $Y$ are polish, then the product topology $X \times Y$ is polish.
- More generally, the countable product $\prod_{n<\omega} X_n$ with the product topology is polish if each $X_n$ is polish.
- In particular, the *Baire Space* $\mathcal{N} = \prod_{n<\omega} (\omega, \mathcal{P}(\omega))$ is polish, and so is the *Cantor Space* $\mathcal{C} = \prod_{n<\omega} (2, \mathcal{P}(2))$ (i.e. the countable product of the discrete topology on $2 = \{0, 1\}$).

(Proofs of these are not so interesting for a set theorist. But are in the notes.)
We are most interested in the Baire Space $\mathcal{N}$ and the Cantor Space $\mathcal{C}$. The topologies of these are the result of trees and our analysis will often be the result of looking at trees. As product topologies, the actual open sets are sometimes hard to think about (dealing with rectangles). An easier presentation is below.

We view $\mathcal{N} = \omega \omega$ as infinite branches of $<\omega \omega$. The basic open sets of $\mathcal{N}$ are the result of cones in $<\omega \omega$. Open sets are then unions of cones.
**Definition**

Let $\sigma \in \langle \omega \omega \rangle$. The *cone* $\mathcal{N}_\sigma$ is the set

$$\mathcal{N}_\sigma = \{ x \in \omega \omega : \sigma \subseteq x \}.$$  

The topology on $\mathcal{N} = \omega \omega$ is the one generated by cones.

One can show that this topology is the same as the product topology and also is homeomorphic to the standard topology on $\mathbb{R} \setminus \mathbb{Q}$. 
The Cantor Space has a similar presentation but instead it’s infinite binary sequences $\mathcal{C} = \omega_2$ coming from the tree of finite binary sequences $\prec \omega_2$.

Notice that $\mathcal{N}$ is not homeomorphic to $\mathbb{R}$’s standard topology: the only clopen subsets of $\mathbb{R}$ are the obvious ones: $\mathbb{R}$ and $\emptyset$. These cones $\mathcal{N}_\sigma$ are all clopen however.

**Proof.**

Each $\mathcal{N}_\sigma$ is open, and to see that it’s closed, we can write the complement

$$\mathcal{N} \setminus \mathcal{N}_\sigma = \bigcup_{\tau \perp \sigma} \mathcal{N}_\tau$$

where $\tau \perp \sigma$ iff $\tau$ and $\sigma$ disagree somewhere (i.e. $\tau$ isn’t an initial segment of $\sigma$ and vice versa).

Going beyond this, $\mathcal{C}$ and $\mathcal{N}$ aren’t homeomorphic either since $\mathcal{C}$ is compact whereas $\mathcal{N}$ isn’t.
If open sets are the result of cones, i.e. decided by a finite amount of information, what do the closed sets of $\mathcal{N}$ (and $\mathcal{C}$) look like? We still have the same metric space results about what “closed” means, but the following is a more practical understanding.

**Definition**

Let $X$ be a set. $T \subseteq \omega X$ is a tree over $X$ iff $T$ is closed under initial segments: $\sigma \subseteq \tau \in T$ implies $\sigma \in T$. So $\langle T, \subseteq \rangle$ is a tree in the usual mathematical sense.

For $T$ a tree over $X$, the set of infinite branches of $T$ is denoted

$$[T] = \{x \in \omega X : \forall n \in \omega (x|n \in T)\}.$$

**Theorem**

A set $X \subseteq \mathcal{N}$ is closed iff there is some tree $T$ over $\omega$ such that $X = [T]$. 
The Baire Space

**Theorem**

A set $X \subseteq \mathcal{N}$ is closed iff there is some tree $T \subseteq ^{<\omega}\omega$ such that $X = [T]$.

**Proof.**

- If $T$ is a tree, then $[T]$ is closed because if $x \in \mathcal{N} \setminus [T]$ then there is some initial segment $\tau$ of $x$ not in $T$.
- Clearly no extension of $\tau$ can be in $T$ and thus $\mathcal{N}_\tau \cap [T] = \emptyset$, i.e. $\mathcal{N}_\tau \subseteq \mathcal{N} \setminus [T]$ so that $\mathcal{N} \setminus [T]$ is open.
- If $X$ is closed, the tree of initial segments of elements of $X$

$$T = \{\tau \in ^{<\omega}\omega : \mathcal{N}_\tau \cap X \neq \emptyset\}$$

works. Clearly $X \subseteq [T]$, and the fact that $X$ is closed is what gives $[T] \subseteq X$.
- $x \in [T] \setminus X$ with $X$ closed implies there’s an initial segment $\tau$ with $\mathcal{N}_\tau \cap X = \emptyset$. But $\tau \in T$ implies $\mathcal{N}_\tau \cap X \neq \emptyset$, a contradiction. \(\blacksquare\)
The Baire Space

As before, polish spaces are not homeomorphic to $\mathbb{R}$ or $\mathcal{N}$, but

- $\mathcal{N}$ is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$, cones correspond to intervals;
- $\mathcal{C}$ is homeomorphic to $[0, 1] \setminus \mathbb{Q}$, cones correspond to intervals.

Nevertheless, all polish spaces basically look like $\mathcal{N}$ by the following two results: for $\mathcal{M}$ a polish space,

- There is a continuous surjection $f : \mathcal{N} \to \mathcal{M}$;
- If $\mathcal{M}$ has no isolated points, there is a continuous injection $f : \mathcal{N} \to \mathcal{M}$.

Note that $\mathcal{M}$ can only have countably many isolated points (by separability). Also the above hold for $\mathcal{C}$ in place of $\mathcal{N}$.

So most of the results we care about can be translated through these continuous maps, and hence we can get a lot of results about the “actual” real numbers by instead studying $\mathcal{C}$ and $\mathcal{N}$.

Question: what do continuous maps from $\mathcal{N}$ look like?
The best way to think about continuity is about sequences converging:

- \( f : \mathcal{N} \to \mathcal{M} \) is continuous iff \( f(x) \) can be built up knowing more and more about \( x \in \omega \omega \).
- Stated formally, \( \bigcap_{n<\omega} f'' \mathcal{N}_x \upharpoonright n = \{ f(x) \} \).
- So to confirm that a map is continuous, we just need that if \( x \) and \( y \) agree on a sufficiently large initial segment, \( f(x) \) and \( f(y) \) agree on a large initial segment.
- This may seem unnecessary for our purposes since we’re focusing so much just on \( \mathcal{N} \) and \( \mathcal{C} \), but we will also be working with products of these.
- Also note the polish topology on \( \omega \) is discrete so every subset is open, i.e. every map from \( \omega \) is continuous.

This is useful for confirming that we don’t actually care about coding: if we view the pair \( \langle x, y \rangle \in \mathcal{N}^2 \) instead as

\[
x \ast y = \langle x(0), y(0), x(1), y(1), x(2), y(2), \cdots \rangle \in \mathcal{N},
\]

this won’t upset anything (as we will see) because the map \( \langle x, y \rangle \mapsto x \ast y \) is continuous.
Recall that a \( \sigma \)-algebra is a collection of sets closed under complements and countable unions (and so countable intersections).

**Definition**

For a topological space \( \mathcal{M} \), the *Borel \( \sigma \)-algebra* \( \mathcal{B}^{\mathcal{M}} \) of \( \mathcal{M} \) is the \( \subseteq \)-least \( \sigma \)-algebra containing the open sets of \( \mathcal{M} \).

Note that every borel set has a “reason” for being borel: we can consider it as the result of a countable union of borel sets, or the complement of some other borel set. In this way, we can build up the borel sets stage by stage using these operations.
Defining the Borel Hierarchy

- We start with the collection of open sets (of $\mathcal{N}$): $\Sigma^0_1$.
- $X \subseteq \mathcal{N}$ is in $\Pi^0_\alpha$ iff $\mathcal{N} \setminus X \in \Sigma^0_\alpha$.
- $X \subseteq \mathcal{N}$ is in $\Sigma^0_{\alpha+1}$ iff $X = \bigcup_{n<\omega} Y_n$ for various $Y_n \in \Pi^0_\alpha$.
- $\Delta^0_\alpha = \Sigma^0_\alpha \cap \Pi^0_\alpha$.

For limit $\alpha$, we can generalize the definition of $\Sigma^0_{\alpha+1}$ by instead just by saying

- $X \subseteq \mathcal{N}$ is $\Sigma^0_\alpha$ for $\alpha > 1$ iff $X = \bigcup_{n<\omega} Y_n$ for some $\{Y_n : n < \omega\} \subseteq \bigcup_{\xi < \alpha} \Pi^0_\xi$.

\[
\begin{align*}
\Delta^0_1 & \subseteq \Pi^0_1 & \Sigma^0_1 & \subseteq \Pi^0_2 & \Sigma^0_2 & \cdots & \Sigma^0_\omega & \subseteq \Pi^0_\omega \subseteq \Delta^0_\omega_1 = \mathcal{B}
\end{align*}
\]

Countable unions are in blue.
Complements are in red.
We start with the collection of open sets (of $\mathcal{N}$): $\Sigma^0_1$.

- $X \subseteq \mathcal{N}$ is in $\Pi^0_\alpha$ iff $\mathcal{N} \setminus X \in \Sigma^0_\alpha$.
- $X \subseteq \mathcal{N}$ is in $\Sigma^0_\alpha$ (for $\alpha > 1$) iff $X = \bigcup_{n<\omega} Y_n$ for some $\{Y_n : n < \omega\} \subseteq \bigcup_{\xi<\alpha} \Pi^0_\xi$.
- $\Delta^0_\alpha = \Sigma^0_\alpha \cap \Pi^0_\alpha$.

These $\Sigma^0_\alpha$ etc. are called *pointclasses* in that they are in $\mathcal{P}(\mathcal{P}(\mathcal{N}))$. It’s not difficult to see that $\Sigma^0_{\omega_1}$ will indeed be a $\sigma$-algebra because of these containments and the fact that $\text{cof}(\omega_1) > \omega$. Showing these containments (the non-strict versions) isn’t too hard.
**Theorem**

For $1 \leq \alpha < \beta$,

1. $\Sigma^0_1 \subseteq \Sigma^0_2$;
2. $\Sigma^0_\alpha \subseteq \Sigma^0_\beta$ and similarly for $\Pi$;
3. $\Delta^0_\alpha \subseteq \Sigma^0_\alpha \subseteq \Delta^0_{\alpha+1}$, and similarly for $\Pi$.

(1) is basically the base case for (2) and might fail in some (weird) topological spaces.

**Proof of (1) & (2).**

1. If $X$ is open, $X$ is the union of a bunch of cones. There are only countably many cones so that $X$ is actually the countable union of a bunch of clopen (and hence $\Pi^0_1$) sets and therefore $X \in \Sigma^0_2$.
2. We just proved when $\alpha = 1$, $\beta = 2$. For $\alpha > 1$, the result is trivial: if $X$ is the countable union $\bigcup_{n<\omega} X_n$ of sets appear in $\Pi^0_\xi$'s for $\xi$'s less than $\alpha$, then these $\xi$'s are also less than $\beta$. 
Theorem

For \( 1 \leq \alpha \),

3. \( \Delta_\alpha^0 \subseteq \Sigma_\alpha^0 \subseteq \Delta_{\alpha+1}^0 \), and similarly for \( \Pi \).

Proof of (3).

3. \( \Delta_\alpha^0 \subseteq \Sigma_\alpha^0 \) is obvious by definition: \( \Delta_\alpha^0 = \Sigma_\alpha^0 \cap \Pi_\alpha^0 \). For \( \Sigma_\alpha^0 \subseteq \Delta_{\alpha+1}^0 \), we have by (2) that \( \Sigma_\alpha^0 \subseteq \Sigma_{\alpha+1}^0 \) so we just need \( \Sigma_\alpha^0 \subseteq \Pi_{\alpha+1}^{0} \). But any \( X \in \Sigma_\alpha^0 \) has \( \mathcal{N} \setminus X \in \Pi_\alpha^0 \). Thus taking the “countable union” of just this one set yields \( \mathcal{N} \setminus X \in \Sigma_{\alpha+1}^0 \) and taking the complement again yields \( X \in \Pi_{\alpha+1}^0 \). Thus \( \Sigma_\alpha^0 \subseteq \Sigma_{\alpha+1}^0 \cap \Pi_{\alpha+1}^0 = \Delta_{\alpha+1}^0 \). \( \dashv \)
Often it’s useful to classify sets in the Borel hierarchy or at least get some sort of bound. In doing so, it’s nice to have some closure properties to realize we don’t need to regard finite unions as countable unions which increase complexity:

\[
\begin{align*}
A & \quad \cup \quad B \\
\Pi_0^1 & \quad \approx_1 \quad \Pi_0^1 \\
\text{naïvely } \Sigma_2^0 &
\end{align*}
\]

\[
\begin{align*}
A & \quad \cup \quad B \\
\Pi_0^1 & \quad \approx_1 \quad \Pi_0^1 \\
\Pi_0^1 &
\end{align*}
\]

All the borel pointclasses are closed under
- finite unions, finite intersections, and continuous preimages.

They are closed under continuous preimages in the sense that for any polish spaces \( \mathcal{M} \) and \( \mathcal{W} \) and continuous \( f : \mathcal{M} \to \mathcal{W} \), if \( X \in \Sigma_0^\alpha,\mathcal{W} \) then \( f^{-1}''X \in \Sigma_0^\alpha,\mathcal{M} \) and similarly for \( \Pi \) and \( \Delta \).
Closure Properties of the Borel Pointclasses

Result

*Every borel pointclass is closed under continuous preimages.*

Proof.

Holds for $\Sigma^0_1$ holds by definition of continuity. Preimages play nicely with complements and unions, so the result holds inductively.

Another way of stating closure under continuous preimages is instead through properties and relations: if $R \subseteq \mathcal{N}$ is “simple” (open, $\Pi^0_\alpha$, etc.) and $f : \mathcal{M} \to \mathcal{N}$ is “simple” (continuous), the relation $P$ given by

$$P(x) \iff R(f(x))$$

is also “simple” (open, $\Pi^0_\alpha$, etc.).

This is why things (measurable, continuous, ergodic, etc.) are often stated in terms of preimages: the relation defined by substituting in the function doesn’t affect the complexity. Hence through (continuous) coding we often refer to relations (as subsets of $\mathcal{N}^{<\omega}$) as Borel despite usually only studying the Borel subsets of $\mathcal{N}$.
Closure Properties of the Borel Pointclasses

With this “logical” perspective,

- continuous preimages correspond to substitution of continuous functions;
- finite intersections and unions correspond to $\land$ and $\lor$ between predicates;
- complements corresponds to negation;
- countable union corresponds to existential quantification over $\omega$

$(x \in \bigcup_{n<\omega} X_n \iff \exists n \in \omega \ R(x, n) \text{ for } R \subseteq \mathcal{N} \times \omega).$

We can make these correspondences more precise later, but it’s motivating for the intuition.

It also motivates notation:

$$\neg \Sigma^0_\alpha = \{ \mathcal{N} \setminus X : X \in \Sigma^0_\alpha \} = \Pi^0_\alpha,$$

for example. Closure under finite intersections says

$$\Sigma^0_\alpha \land \Sigma^0_\alpha = \{ X \cap Y : X \in \Sigma^0_\alpha \land Y \in \Sigma^0_\alpha \} \subseteq \Sigma^0_\alpha.$$
Closure Properties of the Borel Pointclasses

Some other closure properties of the borel pointclasses:

**Result**

All borel pointclasses are closed under finite unions & intersections.

1. Each $\Sigma^0_\alpha$ is closed under countable unions;
2. Each $\Pi^0_\alpha$ is closed under countable intersections;
3. Each $\Delta^0_\alpha$ is closed under complements.

**Proof.**

- Countable unions are easy: $\bigcup_{n<\omega} \bigcup_{m<\omega} X_{n,m} = \bigcup_{n,m<\omega} X_{n,m}$.
- For finite intersections, this holds for open sets, and for $\alpha > 1$,
  $$\bigcup_{n<\omega} X_n \cap \bigcup_{m<\omega} Y_m = \bigcup_{n,m<\omega} \underbrace{X_m \cap Y_n}_{\text{inductively } \Pi^0_{<\alpha}}.$$

- Countable intersections for $\Pi^0_\alpha$ follows from De Morgan’s laws.
- $-\Delta^0_\alpha = - (\Sigma^0_\alpha \cap \Pi^0_\alpha) = -\Sigma^0_\alpha \cap -\Pi^0_\alpha = \Pi^0_\alpha \cap \Sigma^0_\alpha$. 


Closure Properties of the Borel Pointclasses

All of this tells us that $\mathcal{B} = \bigcup_{\alpha < \omega_1} \Sigma^0_\alpha = \bigcup_{\alpha < \omega_1} \Sigma^0_\alpha$ is indeed a σ-algebra and in fact the Borel sets (i.e. the least σ-algebra containing $\Sigma^0_1$).

**Result**

$\bigcup_{\alpha < \omega_1} \Sigma^0_\alpha$ is a σ-algebra and hence $\Sigma^0_\alpha = \Delta^0_\alpha = \Pi^0_\alpha = \Sigma^0_{\omega_1}$ for $\alpha \geq \omega_1$.

**Proof.**

- We know $\bigcup_{\alpha < \omega_1} \Sigma^0_\alpha$ is closed under complements since $-\Sigma^0_\alpha = \Pi^0_\alpha \subseteq \Sigma^0_{\alpha+1}$.
- For countable unions, if $\{X_n : n \in \omega\} \subseteq \bigcup_{\alpha < \omega_1} \Sigma^0_\alpha$, then $X_n \in \bigcup_{\alpha_n < \omega_1} \Sigma^0_{\alpha_n} \subseteq \Pi^0_{\alpha+n+1}$ for $\alpha_n < \omega_1$.
- Since $\text{cof}(\omega_1) > \omega$, these $\alpha_n + 1$s are bounded by some $\alpha$: $\Pi^0_{\alpha_n+1} \subseteq \Pi^0_{\alpha} \subseteq \Sigma^0_{\alpha+1}$ so that $\{X_n : n \in \omega\} \subseteq \Sigma^0_{\omega+1}$.
- Closure under countable unions gives $\bigcup_{n \in \omega} X_n \in \Sigma^0_{\alpha+1} \subseteq \bigcup_{\xi < \omega_1} \Sigma^0_\xi$.  \[\blacksquare\]
How do we show all the containments below are strict?

\[
\Delta^0_1 \cup \Sigma^0_1 \cup \Pi^0_1 \subsetneq \Delta^0_2 \cup \Sigma^0_2 \cup \Pi^0_2 \\
\Delta^0_3 \cup \Sigma^0_3 \cup \Pi^0_3 \subsetneq \cdots \subsetneq \Delta^0_\omega \cup \Sigma^0_\omega \cup \Pi^0_\omega \\
\Delta^0_\omega+1 \cup \Sigma^0_\omega+1 \cup \Pi^0_\omega+1 \subsetneq \cdots \subsetneq \Delta^0_{\omega+1} = \mathcal{B}
\]

We use the concept of a “universal set” in that its slices list out all of the sets of the pointclass.

**Definition**

Let $\Gamma$ be a Borel pointclass. A set $U \subseteq \mathcal{N} \times \mathcal{N}$ is $\Gamma$-universal iff $U \in \Gamma$ (or rather the version of $\Gamma$ in the product $\mathcal{N} \times \mathcal{N}$) and for every $A \in \Gamma$, there is an $r \in \mathcal{N}$ where

\[
A = U_r = \{ x \in \mathcal{N} : \langle x, r \rangle \in U \}.\]
Definition

Let $\Gamma$ be a Borel pointclass. A set $U \subseteq \mathcal{N} \times \mathcal{N}$ is $\Gamma$-universal iff $U \in \Gamma$ and for every $A \in \Gamma$, there is an $r \in \mathcal{N}$ where

$$A = U_r = \{x \in \mathcal{N} : \langle x, r \rangle \in U\}.$$

Proving that every $\Sigma^0_0$ and $\Pi^0_0$ has a universal set isn’t terribly interesting, just having to code some things. Assuming they exist, we get the following.

Theorem

For all $\alpha < \omega_1$, $\Sigma^0_\alpha \neq \Pi^0_\alpha$ and all of the containments of the Borel hierarchy are strict.
Theorem

For all \( \alpha < \omega_1 \), \( \Sigma^0_{\alpha} \not\subseteq \Pi^0_{\alpha} \) and all of the containments of the Borel hierarchy are strict.

Proof.

- If \( U \) is \( \Sigma^0_{\alpha} \)-universal, consider \( D = \{ x : \langle x, x \rangle \in U \} \) which is in \( \Sigma^0_{\alpha} \) as the continuous preimage of \( x \mapsto \langle x, x \rangle \).
- \( D \not\in \Pi^0_{\alpha} \) since if it were, then \( \mathcal{N} \setminus D \in \Sigma^0_{\alpha} \).
- So \( \mathcal{N} \setminus D = U_r \) for some \( r \).
- Is \( r \in \mathcal{N} \setminus D \)? If so, \( \langle r, r \rangle \notin U \) i.e. \( r \notin U_r = \mathcal{N} \setminus D \).
- If not, \( r \in D \) so \( \langle r, r \rangle \in U \) and thus \( r \in U_r = \mathcal{N} \setminus D \), a contradiction. Thus \( U, D \in \Sigma^0_{\alpha} \setminus \Pi^0_{\alpha} \).
- A similar idea yields a member of \( \Pi^0_{\alpha} \setminus \Sigma^0_{\alpha} \).
- Hence \( \Delta^0_{\alpha} \not\subseteq \Sigma^0_{\alpha} \). We also have any \( \Pi^0_{\alpha} \)-universal set is in \( \Delta^0_{\alpha+1} \setminus \Sigma^0_{\alpha} \) so \( \Sigma^0_{\alpha} \not\subseteq \Delta^0_{\alpha+1} \).
A Summary

For \( \Gamma \) an arbitrary borel pointclass,

\[
\begin{align*}
\Gamma \vee \bar{\Gamma} &= \bar{\Gamma} \\
\Gamma \wedge \bar{\Gamma} &= \Gamma \\

\neg \Sigma &= \Pi \\
\bigvee_{i < \omega} \Sigma^i &= \Sigma \\
\bigwedge_{i < \omega} \Pi^i &= \Pi \\

\neg \Delta^0 &= \Delta
\end{align*}
\]
One natural question to ask is the following:

- To what extent do the Borel sets determine the topology?
- In other words, does $\mathcal{B}$ determine $\Sigma^0_1$?

The answer is an emphatic *not at all!* (Proof of the following is in notes.)

**Theorem**

Let $\tilde{M} = \langle M, \mathcal{O} \rangle$ be a polish topological space with $X \in \mathcal{B}^M$. Therefore there is an $\mathcal{O}' \supseteq \mathcal{O}$ such that

- $\tilde{M}' = \langle M, \mathcal{O}' \rangle$ is a polish topological space;
- $\mathcal{B}^{\tilde{M}} = \mathcal{B}^{\tilde{M}'}$; and
- $X \in \Delta^0_1$ is clopen in $\tilde{M}'$.

This theorem has various nice consequences because if we can always prove something for open sets of arbitrary polish spaces and these properties are preserved under things like continuous preimages, we often can prove the result for *all* borel sets by changing the topology in this way.
An analogous question to ask is

- How do we know the borel hierarchies on \( \mathcal{N}, \mathcal{C}, \mathbb{R}, \mathbb{R}^n \), and other polish spaces are similar?
- How can we translate theorems about \( \mathcal{N} \) or \( \mathcal{C} \) to theorems about \( \mathbb{R} \)?

We cannot in general identify \( \Sigma^0_\alpha \) on \( \mathcal{N} \) with \( \Sigma^0_\alpha \) on an arbitrary polish space \( \mathcal{M} \) since we can change the topology too much. That said, there is still a uniqueness to the borel sets.

**Theorem (Borel Isomorphism Theorem)**

Let \( \mathcal{M} \) be an uncountable polish space. Therefore there is a bijection \( f : \mathcal{N} \to \mathcal{M} \) such that

- the preimage of any Borel set is Borel; and
- the image of any Borel set is Borel.

In particular, \( f^{-1}(\mathcal{B}) = \mathcal{B}^\mathcal{M} \). Moreover, we can calculate the “complexity” of these \( f \) in that \( f^{-1}(\Sigma^0_\alpha) = \Sigma^0_{\beta + \alpha} \) for some \( \beta \) (in particular, the \( \beta \) where \( f^{-1}(\Sigma^1_1) = \Sigma^0_\beta \)). Hence the two borel hierarchies will indeed act the same.
Defining the Projective Hierarchy

The Borel hierarchy is one of two “boldface” hierarchies, the other being the *projective* hierarchy.

- The ‘0’ in $\Sigma^0_\alpha$ basically represents quantification over $\omega$: countable unions are $\exists n \in \omega$.
- If we change this ‘0’ to a ‘1’, we get quantification over $\mathcal{P}(\mathbb{N}) \approx \mathcal{N}$.
- We can go further than this to $\mathcal{P}(\mathcal{P}(\mathbb{N}))$, and so on in higher order logics.

The relevant operation for the Borel hierarchy was countable union. For the projective hierarchy, it’s projection.

**Definition**

For $A \subseteq X \times Y$ (usually $X, Y$ polish spaces), the *projection* of $A$, $\pi A$ is just $\text{dom}(A)$:

$$\pi A = \{x \in X : \exists y \in Y \langle x, y \rangle \in A\} \subseteq X.$$  

We write $\pi_X A$ if there’s an ambiguity like with $A \subseteq X \times Y \times Z$.

In principle, we also could talk about projection from coding instead of from genuine product spaces, e.g. $\pi A = \{x : \exists y (x * y \in A)\}$.  


Defining the Projective Hierarchy

**Definition**

For $X \subseteq \mathcal{N}$, $n < \omega$;

- $X$ is $\Sigma^1_0$ iff $X$ is $\Sigma^0_1$ i.e. open;
- $X$ is $\Sigma^1_{n+1}$ iff $X = \mathcal{P}A$ for some $A \in \Pi^1_n$;
- $X$ is $\Pi^1_n$ iff $\mathcal{N} \setminus X$ is $\Sigma^1_n$;
- $X$ is $\Delta^1_n$ iff $X \in \Sigma^1_n \cap \Pi^1_n$.

These are the projective pointclasses, and such $X$ are *projective*.

The general picture we get form the projective hierarchy is the same sort of argyle pattern as before, this time of length $\omega$:

\[\Sigma^0_1 = \Sigma^1_0\]
\[\Pi^0_1 = \Pi^1_0\]

\[\mathcal{B} = \Delta^1_1 \cup \Sigma^1_2 \cup \Pi^1_2 \cup \Delta^1_3 \cup \ldots\]
Defining the Projective Hierarchy

**Definition**

For \( X \subseteq \mathcal{N}, n < \omega \);

- \( X \) is \( \Sigma^1_0 \) iff \( X \) is \( \Sigma^0_1 \) i.e. open;
- \( X \) is \( \Sigma^1_0 \) iff \( X = pA \) for some \( A \in \Pi^1_n \);
- \( X \) is \( \Pi^1_n \) iff \( \mathcal{N} \setminus X \) is \( \Sigma^1_n \);
- \( X \) is \( \Delta^1_n \) iff \( X \in \Sigma^1_n \cap \Pi^1_n \).

These are the projective pointclasses, and such \( X \) are *projective*.

The way it’s generated is also similar:

\[ \Sigma^0_1 = \Sigma^0_1 \]
\[ \Pi^0_1 = \Pi^0_1 \]
\[ \Sigma^1_1 = \Delta^1_1 \]
\[ \Pi^1_1 = \Pi^1_1 \]
\[ \Sigma^1_2 \]

**Projections are in blue**

**Complements are in red**
As I’ve defined things here, $\mathcal{p}\Pi^0_1 = \Sigma^1_1$. But there are about a million different ways to define $\Sigma^1_1$ sets.

In particular, the following are all equivalent:

- $X \subseteq \mathcal{N}$ is $\Sigma^1_1$;
- $X = \mathcal{p}Y$ for some closed $Y \subseteq \mathcal{N} \times \mathcal{N}$;
- $X = \{x : \exists y \ (x \ast y \in Y)\}$ for some closed $Y \subseteq \mathcal{N}$;
- $X = \mathcal{p}Y$ for some Borel $Y \subseteq \mathcal{N} \times \mathcal{N}$;
- $X = \mathcal{p}Y$ for some closed $Y \subseteq \mathcal{N} \times \mathcal{M}$ where $\mathcal{M}$ is some polish space;
- $X = \text{im } f$ for some continuous $f : \mathcal{N} \rightarrow \mathcal{N}$.

Showing all of these equivalences takes some time and relies on closure properties of $\Sigma^1_1$-sets and the other projective pointclasses more generally.
The projective pointclasses are closed under the usual logical operations.

**Theorem**

For $0 < n < \omega$,

- $\Sigma^1_n$ is closed under countable unions, countable intersections, and projections.
- $\Pi^1_n$ is closed under countable unions, countable intersections, and co-projections (corresponding to $\forall$)
- $\Delta^1_n$ is closed under countable unions and complements, and is thus a $\sigma$-algebra.

The real meat of this theorem is the base case of the induction: showing $\Sigma^1_1$ is closed under countable unions and intersections. The rest follows pretty easily.
Result

$\Sigma^1_1$ is closed under countable unions and intersections.

Proof.

Work over $\mathbb{R}$ for the sake of notation. For the countable union $\bigcup_{n<\omega} pY_n$, $Y_n \subseteq \mathbb{R}^2$ closed, consider taking rectangles $Y_n \times [n, n+0.5]$.

- These rectangles $Y_n \times [n, n+0.5]$ are also closed.
- We’ve basically taken the “separated union” of the $Y_n$ so that $\bigcup_{n<\omega} Y_n \times [n, n+0.5]$ is closed.
- Since $p_{\mathbb{R}^2} Y_n \times [n, n+0.5] = Y_n$, we get $p_{\mathbb{R}} Y_n \times [n, n+0.5] = pY_n$ and thus

  \[ p \bigcup_{n<\omega} Y_n \times [n, n+0.5] = \bigcup_{n<\omega} p(Y_n \times [n, n+0.5]) = \bigcup_{n<\omega} pY_n \in \Sigma^1_1. \]

For countable intersections, we just use a nice coding (see 16 C • 2 in the notes).
The last closure property we care about for projective pointclasses is closure under *Borel* substitution in the sense of closure under Borel preimages.

**Definition**

For $\mathcal{M}$, $\mathcal{W}$ two topologies, a function $f : \mathcal{M} \to \mathcal{W}$ is *Borel* iff for any $Y \in \mathcal{B}^{\mathcal{W}}$, $f^{-1}Y \in \mathcal{B}^{\mathcal{M}}$.

- This is clearly a weakening of being continuous.
- So all continuous functions are borel, but there are many borel functions that are not continuous.
- The point is that we don’t increase complexity in the projective hierarchy when taking preimages or substituting Borel functions: if $R$ defines a projective set and $f$ is Borel, $P$ defined by $P(x) \leftrightarrow R(f(x))$ defines a projective set as well with the same complexity as $R$.
- We also get that we’re closed under Borel *images*, but we don’t prove this.
The only real things left about the basics of the projective hierarchy are

- \( \Delta^1_1 = \mathcal{B} \), i.e. the borel sets are precisely the sets that are both \( \Sigma^1_1 \) and \( \Pi^1_1 \).
- the argyle containments (and that they’re strict).

The proof that \( \Delta^1_1 = \mathcal{B} \) uses some nice ideas and techniques, but is too technical and boring for us. (Proof is 16 C • 8 in the notes.)

**Result**

For \( n < \omega \), \( \Delta^1_n \subseteq \Sigma^1_n \subseteq \Delta^1_{n+1} \) and similarly for \( \Pi \).

The proof of this is a little easier than with the Borel hierarchy assuming the closure properties of the projective pointclasses.
Containments of the Projective Pointclasses

Result

For $n < \omega$, $\Delta^1_n \subseteq \Sigma^1_n \subseteq \Delta^1_{n+1}$ and similarly for $\Pi$.

Proof.

- $\Delta^1_n \subseteq \Sigma^1_n$ by definition.
- Note that closed sets are in $\Sigma^1_1$ since $U \times \mathcal{N}$ with $U \subseteq \mathcal{N}$ closed has $p(U \times \mathcal{N}) = U$ as the projection of a closed set.
- For $n = 0$, $\Sigma^1_1$ contains all closed sets and is closed under countable unions: $\Sigma^0_2 \subseteq \Sigma^1_1$ and $\Sigma^0_2 \subseteq \Sigma^0_1$. Thus $\Sigma^1_0 = \Sigma^0_1 \subseteq \Sigma^1_1$.
- Since the closed sets are $\Sigma^1_1$, the open sets are all in $\Pi^1_1$.
- Therefore $\Sigma^1_0 = \Sigma^0_1 \subseteq \Sigma^1_1 \cap \Pi^1_1 = \Delta^1_1$.
- For $n > 0$, inductively $\Pi^1_{n-1} \subseteq \Pi^1_n$. Taking projections yields $\Sigma^1_n \subseteq \Sigma^1_{n+1}$.
- $\Sigma^1_n \subseteq \Pi^1_{n+1}$ because $\Pi^1_n \subseteq \Sigma^1_{n+1}$. This is because $A \in \Pi^1_n$ has $A \times \mathcal{N} \in \Pi^1_n$ and $A = p(A \times \mathcal{N}) \in \Sigma^1_{n+1}$. 

1
Containments of the Projective Pointclasses

That the containments below are strict follows in the exact same way as with the Borel hierarchy: we find universal sets for each $\Sigma^1_1$ and each $\Pi^1_1$ and these universal sets show the strict containments.

- $U$ a $\Sigma^1_1$-universal set gives $U \in \Sigma^1_n \setminus \Delta^1_n$.
- $W$ a $\Pi^1_1$-universal set gives $U \in \Delta^1_{n+1} \setminus \Sigma^1_n$.

\[
\Sigma^0_1 = \Sigma^1_0 \\
\Pi^0_1 = \Pi^1_0 \\
\Sigma^1_1 \subseteq \Delta^1_1 \\
\Pi^1_1 \subseteq \Delta^1_2 \\
\Sigma^1_2 \subseteq \Delta^1_3 \\
\ldots
\]
Writing $\Gamma \land \Gamma = \{X \cap Y : X, Y \in \Gamma\}$, $\exists^N \Gamma = p\Gamma = \{pX : X \in \Gamma\}$ and similarly with the other operations. For $\Gamma$ an arbitrary borel or projective pointclass and $P$ an arbitrary projective pointclass,

\[
\begin{align*}
\Gamma \lor \Gamma &= \Gamma \\
\bigvee_{i<\omega} \Sigma &= \Sigma \\
\Gamma \land \Gamma &= \Gamma \\
\bigwedge_{i<\omega} \Pi &= \Pi \\
\exists^N \Sigma^1 &= \Sigma^1 \\
\forall^N \Pi^1 &= \Pi^1 \\
\neg \Sigma &= \Pi \\
\neg \Delta &= \Delta
\end{align*}
\]